Freeness Conditions for Crossed Squares and Squared Complexes.

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Abstract

Following Ellis, [9], we investigate the notion of totally free crossed square and related squared complexes. It is shown how to interpret the information in a free simplicial group given with a choice of CW-basis, interms of the data for a totally free crossed square. Results of Ellis then apply to give a description in terms of tensor products of crossed modules. The paper ends with a purely algebraic derivation of a result of Brown and Loday.
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Introduction

Crossed squares were introduced by Loday and Guin-Walery in [12]. They arose in various problems of relative algebraic K-theory. Loday later showed in [14] that these quite simple algebraic gadgets modelled all homotopy 3-types. More generally his notion of cat^n-group and the related crossed n-cubes of Ellis and Steiner were shown by Loday to model all connected $(n+1)$-types. The possibilities of calculation with these models was enhanced by the development with R.Brown of a van Kampen type theorem for these structures [3].

A link between simplicial groups and crossed n-cubes was used by Porter, [21] to give an algebraic form of Loday's result and in particular to give a functor from the category of simplicial groups to that of crossed n-cubes realising the equivalence.

In 1993, Ellis [9] introduced a notion of free crossed square and showed how to assign a free crossed square to a CW-complex. As there was an established notion of free simplicial group, it seemed important to investigate the extent to which the two notions of freeness are related. That was the initial motivation for this paper. The two notions were intimately related and moreover combining this with Ellis' alternative description of free crossed squares in terms of the Brown-Loday non-abelian tensor product of groups and coproducts of crossed modules, gives a new purely algebraic derivation

of Brown and Loday's result describing the homotopy 3-type of the suspension of an Eilenberg-Mac Lane space. This success raises our hopes that this method of attack can yield new results in higher dimensions.

1 Preliminaries

In this paper we will concentrate on the reduced case and hence on simplicial groups rather than simplicial groupoids. This is for ease of exposition only and all the results do go through for simplicially enriched groupoids.

Notation: If X is a set, $F(X)$ will denote the free group on X. If Y is a subset of $F(X)$, $\langle Y\rangle$ will denote the normal subgroup generated by Y within $F(X)$.

1.1 Simplicial groups and groupoids

Denoting the usual category of finite ordinals by Δ, we obtain for each $k \geq 0$, a subcategory $\Delta_{\leq k}$ determined by the objects $[j]$ of Δ with $j \leq k$. A simplicial group is a functor from the opposite category Δ^{op} to $\mathfrak{Grp}$; a k-truncated simplicial group is a functor from $\Delta^{op}_{\leq k}$ to $\mathfrak{Grp}$. We will denote the category of simplicial groups by $\mathfrak{SimpGrp}$ and the category of k-truncated simplicial groups by $\mathfrak{Tr_k SimpGrp}$. By a *k-truncation of a simplicial group*, we mean a k-truncated simplicial group $\mathfrak{tr_k}\mathbf{G}$ obtained by forgetting dimensions of order $> k$ in a simplicial group $\mathbf{G}$, that is restricting $\mathbf{G}$ to $\Delta^{op}_{\leq k}$. This gives a truncation functor $\mathfrak{tr_k} : \mathfrak{SimpGrp} \longrightarrow \mathfrak{Tr_k SimpGrp}$ which admits a right adjoint $\mathfrak{cosk_k} : \mathfrak{Tr_k SimpGrp} \longrightarrow \mathfrak{SimpGrp}$ called the *k-coskeleton functor*, and a left adjoint $\mathfrak{sk_k} : \mathfrak{Tr_k SimpGrp} \longrightarrow \mathfrak{SimpGrp}$, called the *k-skeleton functor*. For explicit constructions of these see [11]. We will say that a simplicial group G is *k-skeletal* if the natural morphism $\mathfrak{sk_k}G \to G$ is an isomorphism.

Recall that given a simplicial group $\mathbf{G}$, *the Moore complex* (NG, ∂) *of* $\mathbf{G}$ is the normal chain complex defined by

$$(NG)_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n$$

with $\partial_n : NG_n \to NG_{n-1}$ induced from d_n^n by restriction. There is an alternative form of Moore complex given by the convention of taking

$$\bigcap_{i=1}^{n} \mathrm{Ker} d_i^n$$

and using d_0 instead of d_n as the boundary. One convention is used by Curtis [6] (the d_0 convention) and the other by May [15] (the d_n convention). They lead to equivalent theories.

The n^{th} *homotopy group* $\pi_n(\mathbf{G})$ of $\mathbf{G}$ is the n^{th} homology of the Moore complex of $\mathbf{G}$, i.e.

$$\begin{array}{rcl}\pi_n(\mathbf{G}) & \cong & H_n(NG, \partial) \\ & = & \bigcap\limits_{i=0}^{n} \mathrm{Ker} d_i^n / d_{n+1}^{n+1}(\bigcap\limits_{i=0}^{n} \mathrm{Ker} d_i^{n+1}).\end{array}$$

We say that the Moore complex $\mathbf{NG}$ of a simplicial group is of *length* k if $NG_n = 1$ for all $n \geq k+1$, so that a Moore complex of length k is also of length l for $l \geq k$. For example, if $\mathbf{G}$ has Moore complex of length 1, then (NG_1, NG_0, ∂_1) is a crossed module and conversely. If NG is of length 2, the corresponding Moore complex gives a 2-crossed module in the sense of Conduché, [5], cf. the companion paper to this, [20]

1.2 Free Simplicial Groups

Recall from [6] and [13] the definitions of free simplicial group and of a $CW-basis$ for a free simplicial group.

Definition

A simplicial group $\mathbf{F}$ is called *free* if

(a) F_n is a free group with a given basis, for every integer $n \geq 0$,

(b) The bases are stable under all degeneracy operators, i.e., for every pair of integers (i, n) with $0 \leq i \leq n$ and every basic generator $x \in F_n$ the element $s_i(x)$ is a basic generator of F_{n+1}.

Definition

Let $\mathbf{F}$ be a free simplicial group (as above). A subset $\mathfrak{F} \subset \mathbf{F}$ will be called a $CW-basis$ of $\mathbf{F}$ if

(a) $\mathfrak{F_n} = \mathfrak{F} \cap F_n$ freely generates F_n for all $n \geq 0$,

(b) $\mathfrak{F}$ is closed under degeneracies, i.e. $x \in \mathfrak{F_n}$ implies $s_i(x) \in \mathfrak{F_{n+1}}$ for all $0 \leq i \leq n$,

(c) if $x \in \mathfrak{F_n}$ is non-degenerate, then $d_i(x) = e_{n-1}$, the identity element of F_n, for all $0 \leq i < n$.

As explained earlier, we have restricted attention so far to simplicial groups and hence to connected homotopy types. This is traditional but a bit unnatural as all the results and definitions so far extend with little or no trouble to simplicial groupoids in the sense of Dwyer and Kan [7] and hence to non-connected homotopy types. It should be noted that such simplicial groupoids have a fixed and constant simplicial set of objects and so are not merely simplicial objects in the category of groupoids. In this context if $\mathbf{G}$ is a simplicial groupoid with set of objects O, the natural form of the Moore complex $\mathbf{NG}$ is given by the same formula as in the reduced case, interpreting $\mathrm{Ker} d_i^n$ as being the subgroupoid of elements in G_n whose i^{th} face is an identity of G_{n-1}. Of course if $n \geq 1$, the resulting NG_n is a disjoint union of groups, so $\mathbf{NG}$ is a disjoint union of the Moore complexes of the vertex simplicial groups of $\mathbf{G}$ together with the groupoid G_0 providing

elements that allow conjugation between (some of) these vertex complexes (cf. Ehlers and Porter [8]).

Crossed modules of, or over, groupoids are well known from the work of Brown and Higgins. The only changes from the definition for groups (cf. [14]) is that one has to handle the conjugation operation slightly more carefully:

A *crossed module* is a morphism of groupoids $\partial : M \longrightarrow N$ where N is a groupoid with object set O say and M is a family of groups, $M = \{M(a) : a \in O\}$, together with an action of N on M satisfying (i) if $m \in M(a)$ and $n \in N(a,b)$ for $a, b, \in O$, the result of n acting on m is ${}^n m \in M(b)$; (ii) $\partial({}^n m) = n\partial(m)n^{-1}$ and (iii) ${}^{\partial(m)}m' = mm'm^{-1}$ for all $m, m' \in M,\ \ n \in N$. For the weaker notion in which condition (iii) is not required, the models are called *precrossed modules.*

The definition of a CW-basis likewise generalises with each $\mathfrak{F}$ a subgraph of the corresponding free simplicial groupoid.

2 Crossed Squares and Simplicial Groups

Although we will be mainly concerned with crossed squares in this paper, many of the arguments either clearly apply or would seem to apply in the more general case of crossed n-cubes and n-cube complexes. We therefore give some background in this more general setting.

Again although we give the definitions and results for groups, the adaptation to handle groupoids over a fixed base is routine.

The following definition is due to Ellis and Steiner [10]. Let $< n >$ denote the set $\{1, ..., n\}$.

Definition

A *crossed n-cube* of groups is a family $\{\mathfrak{M}_A : A \subseteq < n >\}$ of groups, together with homomorphisms $\mu_i : \mathfrak{M}_A \longrightarrow \mathfrak{M}_{A\backslash\{i\}}$ for $i \in < n >$ and functions

$$h : \mathfrak{M}_A \times \mathfrak{M}_B \longrightarrow \mathfrak{M}_{A\cup B}$$

for $A, B \subseteq < n >$, such that if ${}^a b$ denotes $h(a,b)b$ for $a \in \mathfrak{M}_A$ and $b \in \mathfrak{M}_B$ with $A \subseteq B$, then for all $a, a' \in \mathfrak{M}_A$ and $b, b' \in \mathfrak{M}_B, c \in \mathfrak{M}_C$ and $i, j \in < n >$,

the following hold:

$$\begin{array}{lll}
1) & \mu_i a = a \quad \text{if } i \notin A, & \\
2) & \mu_i \mu_j a = \mu_j \mu_i a, & \\
3) & \mu_i h(a,\ b) = h(\mu_i a,\ \mu_i b), & \\
4) & h(a,\ b) = h(\mu_i a,\ b) = h(a,\ \mu_i b) & \text{if } i \in A \cap B, \\
5) & h(a,\ a') = [a,\ a'], & \\
6) & h(a,\ b) = h(b,\ a)^{-1}, & \\
7) & h(a,\ b) = 1 & \text{if } a = 1 \quad \text{or} \quad b = 1, \\
8) & h(aa',\ b) = {}^a h(a',\ b) h(a,\ b), & \\
9) & h(a,\ bb') = h(a,\ b)\ {}^b h(a,\ b'), & \\
10) & {}^a h(b,\ c) = h({}^a b,\ {}^a c) & \text{if } A \subseteq B \cap C, \\
11) & {}^a h(h(a^{-1},\ b),\ c)\ {}^c h(h(c^{-1},\ a),\ b)\ {}^b h(h(b^{-1},\ c),\ a) = 1. &
\end{array}$$

A morphism of crossed n-cubes is defined in the obvious way: It is a family of group homomorphisms, for $A \subseteq < n >$, $f_A : \mathfrak{M}_A \longrightarrow \mathfrak{M}'_A$ commuting with the μ_i's and h's. We thus obtain a category of crossed n-cubes which will be denoted by $\mathfrak{Crs}^{\mathfrak{n}}$, cf. Ellis and Steiner [10]. Again there is an obvious variant of this definition for groupoids over a fixed set of objects, O.

Remark: Crossed squares, that is the case $n = 2$, were introduced by Loday and Guin-Walery, [12], but with an apparently different definition. The two notions are however equivalent.

Example 1: For $n = 1$, a crossed 1-cube is the same as a crossed module.

For $n = 2$, one has a crossed 2-cube is a crossed square:

$$\begin{tikzcd}
\mathfrak{M}_{<2>} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & \mathfrak{M}_{\{1\}} \arrow[d, "\mu_1"] \\
\mathfrak{M}_{\{2\}} \arrow[r, "\mu_2"'] & \mathfrak{M}_{\emptyset}.
\end{tikzcd}$$

Each μ_i is a crossed module, as is $\mu_1\mu_2$. The h-functions give actions and a function

$$h : \mathfrak{M}_{\{1\}} \times \mathfrak{M}_{\{2\}} \longrightarrow \mathfrak{M}_{<2>}.$$

The maps μ_2 also define a map of crossed modules from $(\mathfrak{M}_{<2>}, \mathfrak{M}_{\{2\}}, \mu_1)$ to $(\mathfrak{M}_{<1>}, \mathfrak{M}_{\emptyset}, \mu_1)$. In fact a crossed square can be thought of as a crossed module in the category of crossed modules.

Example 2: Let N_1, N_2 be normal subgroups of a group G. The commutative square diagram of inclusions;

$$\begin{tikzcd}
N_1 \cap N_2 \arrow[r, "inc."] \arrow[d, "inc."'] & N_2 \arrow[d, "inc."] \\
N_1 \arrow[r, "inc."'] & G
\end{tikzcd}$$

naturally comes together with actions of G on N_1, N_2 and $N_1 \cap N_2$ given by conjugation and functions

$$\begin{array}{rccc} h: & N_A \times N_B & \longrightarrow & N_A \cap N_B = N_{A\cup B} \\ & (n_1, n_2) & \longmapsto & [n_1,\ n_2]. \end{array}$$

That this is a crossed square is easily checked.

The following proposition is noted by the second author in [21].

Proposition 2.1 *[21] Let* $\mathbf{G}$ *be a simplicial group with simplicial normal subgroups* $\mathbf{N_1}$ *and* $\mathbf{N_2}$*. Then the square*

$$\begin{tikzcd} \mathbf{N_1} \cap \mathbf{N_2} \arrow[r] \arrow[d] & \mathbf{N_2} \arrow[d] \\ \mathbf{N_1} \arrow[r] & \mathbf{G} \end{tikzcd}$$

induces a crossed square

$$\begin{tikzcd} \pi_0(\mathbf{N_1} \cap \mathbf{N_2}) \arrow[r] \arrow[d] & \pi_0(\mathbf{N_2}) \arrow[d] \\ \pi_0(\mathbf{N_1}) \arrow[r] & \pi_0(\mathbf{G}). \end{tikzcd}$$

Proof: The h-function

$$h : \pi_0(\mathbf{N_1}) \times \pi_0(\mathbf{N_2}) \longrightarrow \pi_0(\mathbf{N_1} \cap \mathbf{N_2})$$

is given by

$$h(\overline{n_1},\ \overline{n_2}) = \overline{[n_1,\ n_2]}$$

for all $\overline{n_1} \in \pi_0(\mathbf{N_1})$, $\overline{n_2} \in \pi_0(\mathbf{N_2})$. It is then simple, cf. [21], to see that the second diagram above is a crossed square. $\Box$

In fact up to isomorphism all crossed squares arise in this way, cf. [14] and [21].

Example 3: Let $\mathbf{G}$ be a simplicial group. Let $\mathfrak{M}(\mathbf{G}, 2)$ denote the following diagram

$$\begin{tikzcd} NG_2/\partial_3 NG_3 \arrow[r, "\partial_2"] \arrow[d, "\partial_2'"'] & NG_1 \arrow[d, "\mu"] \\ \overline{NG}_1 \arrow[r, "\mu'"'] & G_1 \end{tikzcd}$$

Then this is the underlying square of a crossed square. The extra structure is given as follows: NG_1 =Kerd_0^1 and $\overline{NG}_1$ =Kerd_1^1. Since G_1 acts on $NG_2/\partial_3 NG_3$, $\overline{NG}_1$ and NG_1, there are actions of $\overline{NG_1}$ on $NG_2/\partial_3 NG_3$ and NG_1 via μ', and NG_1 acts on $NG_2/\partial_3 NG_3$ and $\overline{NG}_1$ via μ. Both μ

and μ' are inclusions, and all actions are given by conjugation. The h-map is

$$\begin{array}{ccl} NG_1 \times \overline{NG}_1 & \longrightarrow & NG_2/\partial_3 NG_3 \\ (x,\overline{y}) & \longmapsto & h(x,\ y) = [s_1x,\ s_1ys_0y^{-1}]\partial_3 NG_3. \end{array}$$

Here x and y are in NG_1 as there is a bijection between NG_1 and $\overline{NG}_1$. We leave the verification of the axioms of a crossed square to the reader.
This example is clearly functorial and we denote by

$$\mathfrak{M}(-,2)\ :\ \mathfrak{SimpGrp} \longrightarrow \mathfrak{Crs}^2,$$

the resulting functor. This is the case $n=2$ of a general construction of a crossed n-cube from a simplicial group given by the second author in [21] based on some ideas of Loday.
Examples 2 and 3 revisited: Let G be a group with normal subgroups $N_1,\dots,N_n$ of G. Let

$$\mathfrak{M}_A = \bigcap\{N_i : i \in A\} \quad \text{and} \quad \mathfrak{M}_\emptyset = \mathrm{G}$$

with $A \subseteq < n >$. For $i \in < n >$, $\mathfrak{M}_A$ is a normal subgroup of $\mathfrak{M}_{A-\{i\}}$. Define

$$\mu_i : \mathfrak{M}_A \longrightarrow \mathfrak{M}_{A-\{i\}}$$

to be the inclusion. If $A, B \subseteq < n >$, then $\mathfrak{M}_{A\cup B} = \mathfrak{M}_A \cap \mathfrak{M}_B$, let

$$\begin{array}{rccl} h: & \mathfrak{M}_A \times \mathfrak{M}_B & \longrightarrow & \mathfrak{M}_{A\cup B} \\ & (a,b) & \longmapsto & [\ a,\ b\ ] \end{array}$$

as $[\mathfrak{M}_A, \mathfrak{M}_B] \subseteq \mathfrak{M}_A \cap \mathfrak{M}_B$, where $a \in \mathfrak{M}_A,\ b \in \mathfrak{M}_B$. Then

$$\{\mathfrak{M}_A :\ A \subseteq < n >,\ \mu_i,\ h\}$$

is a crossed n-cube, called the *inclusion crossed n-cube* given by the normal n-ad of groups $(G;\ N_1,\dots,N_n)$.

Proposition 2.2 *Let $(\mathbf{G};\ N_1,\dots,N_n)$ be a simplicial normal n-ad of subgroups of groups and define for $A \subseteq < n >$*

$$\mathfrak{M}_A = \pi_0(\bigcap_{i\in A} N_i)$$

with homomorphisms $\mu_i : \mathfrak{M}_A \longrightarrow \mathfrak{M}_{A-\{i\}}$ and h-maps induced by the corresponding maps in the simplicial inclusion crossed n-cube, constructed by applying the previous example to each level. Then $\{\mathfrak{M}_A :\ A \subseteq < n >,\ \mu_i,\ h\}$ is a crossed n-cube.

Proof: See [21]. $\square$

This describes a functor, [21], from the category of simplicial groups to that of crossed n-cubes of groups.

Theorem 2.3 *If* $\mathbf{G}$ *is a simplicial group, then the crossed* n*-cube* $\mathfrak{M}(\mathbf{G},n)$ *is determined by:*

(i) for $A \subseteq < n >$,

$$\mathfrak{M}(\mathbf{G},n)_A = \frac{\bigcap_{j\in A} \mathrm{Ker} d^n_{j-1}}{d^{n+1}_{n+1}(\mathrm{Ker} d^{n+1}_0 \cap \{\bigcap_{j\in A} \mathrm{Ker}\ d^{n+1}_j\})};$$

(ii) the inclusion

$$\bigcap_{j\in A} \mathrm{Ker} d^n_{j-1} \longrightarrow \bigcap_{j\in A-\{i\}} \mathrm{Ker} d^n_{j-1}$$

induces the morphism

$$\mu_i : \mathfrak{M}(\mathbf{G},n)_A \longrightarrow \mathfrak{M}(\mathbf{G},n)_{A-\{i\}};$$

(iii) the functions, for $A, B \subseteq < n >$,

$$h : \mathfrak{M}(\mathbf{G},n)_A \times \mathfrak{M}(\mathbf{G},n)_B \longrightarrow \mathfrak{M}(\mathbf{G},n)_{A\cup B}$$

are given by

$$h(\bar{x},\bar{y}) = \overline{[x,\ y]},$$

where an element of $\mathfrak{M}(\mathbf{G},n)_A$ *is denoted by* $\bar{x}$ *with* $x \in \bigcap_{j\in A} \mathrm{Ker} d^n_{j-1}$.

□

Some simplification is possible, again see [21] for the details.

Proposition 2.4 *If* $\mathbf{G}$ *is a simplicial group, then*

i) for $A \subseteq < n >,\ A \neq < n >$,

$$\mathfrak{M}(\mathbf{G},n)_A \cong \bigcap_{i\in A} \mathrm{Ker} d^{n-1}_{i-1}$$

so that in particular, $\mathfrak{M}(\mathbf{G},n)_\emptyset \cong G_{n-1}$*; in every case the isomorphism is induced by* d_0,

ii) if $A \neq < n >$ *and* $i \in < n >$,

$$\mu_i : \mathfrak{M}(\mathbf{G},n)_A \longrightarrow \mathfrak{M}(\mathbf{G},n)_{A\backslash\{i\}}$$

is the inclusion of a normal simplicial subgroup,

iii) for $j \in < n >$,

$$\mu_j : \mathfrak{M}(\mathbf{G},n)_{<n>} \longrightarrow \bigcap_{i\neq j} \mathrm{Ker} d^{n+1}_i$$

is induced by d_n.

$\square$

Expanding this data out for low values of n gives:

1) For $n = 0$,

$$\begin{array}{rcl} \mathfrak{M}(\mathbf{G}, 0) & = & G_0/d_1(\mathrm{Ker} d_0,) \\ & \cong & \pi_0(\mathbf{G}), \\ & = & H_0(N\mathbf{G}). \end{array}$$

2) For $n = 1$, $\mathfrak{M}(\mathbf{G}, 1)$ is the crossed module

$$\mu_1 : \mathrm{Ker} d_0^1 / d_2^2(NG_2) \longrightarrow G_1/d_2^2(\mathrm{Ker} d_0^2).$$

3) For $n = 2$, $\mathfrak{M}(\mathbf{G}, 2)$ is

$$\begin{tikzcd} \mathrm{Ker} d_0^2 \cap \mathrm{Ker} d_1^2 / d_3^3(\ \mathrm{Ker} d_0^3 \cap \mathrm{Ker} d_1^3 \cap \mathrm{Ker} d_2^3) \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & \mathrm{Ker} d_0^2 / d_3^3(\mathrm{Ker} d_0^3 \cap \mathrm{Ker} d_1^3) \arrow[d, "\mu_1"] \\ \mathrm{Ker} d_1^2 / d_3^3(\mathrm{Ker}\ d_0^3 \cap \mathrm{Ker} d_2^3) \arrow[r, "\mu_2"'] & G_2/d_3^3(\mathrm{Ker} d_0^3). \end{tikzcd}$$

By Proposition 2.4, this is isomorphic to

$$\begin{tikzcd} NG_2/d_3^3(NG_3) \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & \mathrm{Ker}\ d_0^1 \arrow[d, "\mu_1"] \\ \mathrm{Ker}\ d_1^1 \arrow[r, "\mu_2"'] & G_1, \end{tikzcd}$$

that is

$$\mathfrak{M}(\mathbf{G}, 2) \cong \left(\begin{tikzcd} NG_2/\partial_3(NG_3) \arrow[r] \arrow[d] & \mathrm{Ker}\ d_0 \arrow[d] \\ \mathrm{Ker}\ d_1 \arrow[r] & G_1 \end{tikzcd} \right)$$

is a crossed square. Here the h-map is

$$h : \mathrm{Ker} d_0^1 \times \mathrm{Ker} d_1^1 \longrightarrow NG_2/d_3^3(NG_3)$$

given by $h(x, y) = [s_1 x,\ s_1 y s_0 y^{-1}]\ \partial_3 NG_3$, as before.

Note if we consider the above crossed square as a vertical morphism of crossed modules, we can take its kernel and cokernel within the category of crossed modules. In the above, the morphisms in the top left hand corner are induced from d_2 so

$$\mathrm{Ker}\left(\mu_1 : \frac{NG_2}{\partial_3 NG_3} \longrightarrow \mathrm{Ker} d_1\right) = \frac{NG_2 \cap \mathrm{Ker} d_2}{\partial_3 NG_3} \cong \pi_2(\mathbf{G})$$

whilst the other map labelled μ_1 is an inclusion so has trivial kernel. Hence the kernel of this morphism of crossed modules is

$$\pi_2(\mathbf{G}) \longrightarrow 1.$$

The image of μ_2 is closed and normal in both the groups on the bottom line and as $\mathrm{Ker}d_0 = NG_1$ with the corresponding $\mathrm{Im}\mu_1$ being d_2NG_2, the cokernel is NG_1/∂_2NG_2, whilst $G_1/\mathrm{Ker}d_0 \cong G_0$, i.e., the cokernel of μ_1 is $\mathfrak{M}(\mathbf{G}, 1)$.

In fact of course μ_1 is not only a morphism of crossed modules, it is a crossed module. This means that $\pi_2(\mathbf{G}) \longrightarrow 1$ is in some sense a $\mathfrak{M}(\mathbf{G}, 1)$-module and that $\mathfrak{M}(\mathbf{G}, 2)$ can be thought of as a crossed extension of $\mathfrak{M}(\mathbf{G}, 1)$ by $\pi_2(\mathbf{G})$.

3 Free Crossed Squares

3.1 Definitions

G. Ellis, [9], in 1993 presented the notion of a free crossed square. In this section, we recall his definition and give a construction of free crossed squares by using the second dimensional Peiffer elements and the 2-skeleton of a 'step-by-step' construction of a free simplicial group with given CW-basis. We firstly recall the definition of a free crossed square on a pair of functions (f_2, f_3), as given by Ellis. We will call these crossed squares *totally free.*

Let $\mathbf{B_1}$, $\mathbf{B_2}$ and $\mathbf{B_3}$ be sets. Take $F(\mathbf{B_1})$ to be the free group on $\mathbf{B_1}$. Suppose given a function $f_2 : \mathbf{B_2} \longrightarrow F(\mathbf{B_1})$. Let $\partial : M \longrightarrow F(\mathbf{B_1})$ be the free pre-crossed module on f_2. Using the action of $F(\mathbf{B_1})$ on M we can form the semi-direct product $M \rtimes F(\mathbf{B_1})$. The canonical inclusion $\mu : M \longrightarrow M \rtimes F(\mathbf{B_1})$ given by $m \mapsto (m, 1)$ allows us to consider M as a normal subgroup of $M \rtimes F(\mathbf{B_1})$. (Recall that any normal inclusion is a crossed module with action given by conjugation.) There is a second normal subgroup of $M \rtimes F(\mathbf{B_1})$ arising from M, namely

$$N = \{(m, \partial m^{-1}) : m \in M\} \subset M \rtimes F(\mathbf{B_1})$$

with inclusion denoted $\mu' : N \longrightarrow M \rtimes F(\mathbf{B_1})$. For $m \in M$, we let m' denote the element $(m^{-1},\ \partial m)$ in N.

Assume given a function $f_3 : \mathbf{B_3} \longrightarrow M$, whose image lies in the kernel of the homomorphism $\partial : M \longrightarrow F(\mathbf{B_1})$. There is then a corresponding function $f_3' : \mathbf{B_3} \longrightarrow N$ given by $y \mapsto (f_3(y), 1)$.

Definition [9]

A crossed square,

$$\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, "\partial_2'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & M \rtimes F(\mathbf{B_1}), \end{tikzcd}$$

is totally free on the pair of functions (f_2, f_3) if
(i) $(M, F(\mathbf{B_1}), \partial)$ is the free pre-crossed module on f_2;

(ii) $\mathbf{B_3}$ is a subset of L with f_3 and f_3' the restrictions of ∂_2 and ∂_2' respectively;
(iii) for any crossed square

$$\begin{tikzcd} L' \arrow[r, "\tau"] \arrow[d, "\tau'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & M \rtimes F(\mathbf{B_1}), \end{tikzcd}$$

and any function $\nu : \mathbf{B_3} \longrightarrow L'$ satisfying $\tau\nu = f_3$, there is a unique morphism $\Phi = (\phi, 1, 1, 1)$ of crossed squares:

$$\begin{tikzcd}
L \arrow[rr, "\partial_2"] \arrow[dr, "\phi"] \arrow[dd, "\partial_2'"'] & & M \arrow[drr, "="] \arrow[dd, "\mu"'] & & \\
& L' \arrow[rrr, "\tau"] \arrow[dd, "\tau'"'] & & & M \arrow[dd, "\mu"] \\
N \arrow[rr, "\mu'"] \arrow[dr, "="'] & & M \rtimes F(\mathbf{B_1}) \arrow[drr, "="] & & \\
& N \arrow[rrr, "\mu'"] & & & M \rtimes F(\mathbf{B_1})
\end{tikzcd}$$

such that $\phi\nu' = \nu$, where $\nu' : \mathbf{B_3} \longrightarrow L$ is the inclusion.

We denote such a totally free crossed square by $(L, M, N, M \rtimes F(\mathbf{B_1}))$ omitting the structural morphisms from the notation when there is no danger of confusion.

We know the free pre-crossed module on $f_2 : \mathbf{B_2} \longrightarrow F(\mathbf{B_1})$ is $\partial : \langle \mathbf{B_2} \rangle \longrightarrow F(\mathbf{B_1})$, where $\langle \mathbf{B_2} \rangle$ denotes the normal closure of $\mathbf{B_2}$ in the free group $F(\mathbf{B_2} \cup s_0(\mathbf{B_1}))$, so the function $f_3 : \mathbf{B_3} \longrightarrow M \ \ (= \langle \mathbf{B_2} \rangle)$ is precisely the data $(\mathbf{B_3}, f_3)$ for 2-dimensional construction data in the simplicial context, cf. [19]. We thus need to recall the 2-dimensional construction for a free simplicial group. This 2-dimensional form can be summarised by the diagram

$$\mathbb{F}^{(2)} : \ \ldots\ F(s_1s_0(\mathbf{B_1}) \cup s_0(\mathbf{B_2}) \cup s_1(\mathbf{B_2}) \cup \mathbf{B_3}) \underset{s_1, s_0}{\overset{d_0, d_1, d_2}{\rightrightarrows}} F(s_0(\mathbf{B_1}) \cup \mathbf{B_2}) \underset{s_0}{\overset{d_1, d_0}{\rightleftarrows}} F(\mathbf{B_1})$$

with the simplicial morphisms given as in [19].

3.2 Free crossed squares exist.

Theorem 3.1 *A totally free crossed square $(L, M, N, M \rtimes F(\mathbf{B_1}))$ exists on the 2-dimensional construction data and is given by $\mathfrak{M}(\mathbf{F^{(2)}}, \mathbf{2})$ where $\mathbf{F^{(2)}}$ is the 2-skeletal free simplicial group defined by the construction data.*

Proof: Suppose given the 2-dimensional construction data for a free simplicial group, $\mathbf{F}$, which we will take as above as the data for a totally free crossed square. We will not assume detailed knowledge of [19] so we start with $F(\mathbf{B_1})$ and $f_2 : \mathbf{B_2} \longrightarrow F(\mathbf{B_1})$ and form $M = \langle \mathbf{B_2} \rangle$. This gives $\partial_1 : \langle \mathbf{B_2} \rangle \longrightarrow F(X_0)$ as the free pre-crossed module on f_2. The semidirect product gives

$$F(s_0(\mathbf{B_1}) \cup \mathbf{B_2}) \cong M \rtimes F(\mathbf{B_1})$$

and we can identify this with $\mathbf{F_1^{(2)}}$. This identification also makes

$$M \cong \mathrm{Ker} d_0^1$$

for the d_0^1 of $\mathbf{F}^{(2)}$.

Next form $N = \{(m, \partial m^{-1}) \in M \rtimes F(\mathbf{B_1}) : m \in M\}$. As $m \in \langle \mathbf{B_2} \rangle$, it is a product of conjugates of elements of $\mathbf{B_2}$ and their inverses, so writing $m = \prod (m_{\alpha_i}) y_{\alpha_i}^{\varepsilon_i} (m_{\alpha_i})^{-1}$ for indices α_i, and $\varepsilon_i = \pm 1$, we get $\partial m = \prod m_{\alpha_i} t_{\alpha_i}^{\varepsilon_i} m_{\alpha}^{-1}$ where $t_i = f_2(y_i)$, which is also $d_0^1(y_i)$. Thus we can identify N with $\langle \{ y s_1 d_0^1(y)^{-1} : y \in \mathbf{B_2} \} \rangle$, which is exactly $\mathrm{Ker} d_1^1$.

Now $f_3 : \mathbf{B_3} \to \mathrm{Ker}\partial_1 = \mathrm{Ker}(\partial : NF_1^{(2)} \to NF_0^{(2)}) \subset \langle \mathbf{B_2} \rangle$. We know that this allows us to construct $\mathbf{F}_2^{(2)}$ and hence $\mathbf{F}_n^{(2)}$ for $n \geq 3$, and in addition that taking

$$L = NF_2^{(2)} / \partial_3(NF_3^{(2)}),$$

gives a crossed square

$$\begin{array}{ccc} L & \xrightarrow{\partial} & M \\ {\scriptstyle \partial'} \downarrow & & \downarrow {\scriptstyle \mu} \\ N & \xrightarrow[\mu']{} & F_1^{(2)} \end{array}$$

which is $\mathfrak{M}(\mathbf{F}^{(2)}, 2)$. We claim this is the totally free crossed square on the construction data.

At this stage it is worth noting that there seems to be no simple adjointness statement between $\mathfrak{M}(-,2)$ and some functor that would give a quick proof of freeness. The problem is that $\mathfrak{M}(-,2)$ seems to be an adjoint only up to some sort of coherent homotopy. To avoid this difficulty we use a more combinatorial approach involving the higher dimension Peiffer elements and the explicit description of L.

In [18], we analysed in general the structure of groups of boundaries such as $\partial_3(NF_3^{(2)})$. There we showed that $NF_3^{(2)}$ is normally generated by elements of the following forms:-
(i) For all $x \in NF_1^{(2)},\ y \in NF_2^{(2)}$,

$$\begin{aligned} f_{(1,0)(2)}(x,y) &= [s_1 s_0(x), s_2(y)][s_2(y), s_2 s_0(x)], \\ f_{(2,0)(1)}(x,y) &= [s_2 s_0(x), s_1(y)][s_1(y), s_2 s_1(x)][s_2 s_1(x), s_2(y)][s_2(y), s_2 s_0(x)]; \end{aligned}$$

(ii) for all $y \in NF_2^{(2)}, x \in NF_1^{(2)}$,

$$f_{(0)(2,1)}(x,y) \;=\; [s_0(x), s_2s_1(y)][s_2s_1(y), s_1(x)][s_2(x), s_2s_1(y)],$$

and (iii) for all $x, y \in NF_2^{(2)}$,

$$\begin{array}{rcl} f_{(0)(1)}(x,y) & = & [s_0(x), s_1(y)][s_1(y), s_1(x)][s_2(x), s_2(y)], \\ f_{(0)(2)}(x,y) & = & [s_0(x), s_2(y)], \\ f_{(1)(2)}(x,y) & = & [s_1(x), s_2(y)][s_2(y), s_2(x)]. \end{array}$$

Given our description of $NF^{(2)}$ in low dimensions, it is routine to calculate normal generators of the various groups involved here in terms of $\mathbf{B_1}$ and $\mathbf{B_2}$. We set

$$Z = \{s_1(y)^{-1}s_0(y) : y \in \mathbf{B_2}\}.$$

The above diagram can then be realised as

$$\begin{tikzcd} J \arrow[r, "\partial_2"] \arrow[d, "\partial_2'"'] & \langle \mathbf{B_2} \rangle \arrow[d, "\mu"] \\ \langle Z \rangle \arrow[r, "\mu'"'] & \langle \mathbf{B_2} \rangle \rtimes F(\mathbf{B_1}) \end{tikzcd} .$$

Here J is $(\langle s_1(\mathbf{B_2}) \cup \mathbf{B_3} \rangle \cap \langle Z \cup \mathbf{B_3} \rangle)/P_2$, P_2 being the second dimensional Peiffer normal subgroup, which is in fact just $\partial_3(NF_3^{(2)})$, and which is a subgroup of $\langle s_1(\mathbf{B_2}) \cup \mathbf{B_3} \rangle \cap \langle Z \cup \mathbf{B_3} \rangle$.

Given any crossed square $(L', M, N, M \rtimes F(\mathbf{B_1}))$ and a function $\nu : \mathbf{B_3} \longrightarrow L'$, there then exists a unique morphism

$$\phi : (L, M, N, M \rtimes F(\mathbf{B_1})) \longrightarrow (L', M, N, M \rtimes F(\mathbf{B_1}))$$

given by

$$\phi(y_i' P_2) = \nu(y_i')$$

such that $\phi\nu' = \nu$. The existence of ϕ follows by using the freeness property of the group $NF_2^{(2)}$ and then restricting to $\langle s_1(\mathbf{B_2}) \cup \mathbf{B_3} \rangle \cap \langle Z \cup \mathbf{B_3} \rangle$. The normal generating elements of P_2 are then easily shown to have trivial image in L' as that group is part of the second crossed square.

Thus the diagram is the desired totally free crossed square on the 2-dimensional construction data. The crossed square properties of $(L, M, N, M \rtimes F(\mathbf{B_1})$ may be easily verified or derived from the fact that this is exactly $\mathfrak{M}(\mathbf{F}^{(2)}, 2)$. $\square$

Remark:

At this stage, it is important to note that nowhere in the argument was use made of the freeness of the 1-skeleton. If G is any 1-skeletal simplicial group and we form a new simplicial group H by adding in a set $\mathbf{B_3}$ of new

generators in dimension 2, so that for instance, $H_2 = G_2 * F(\mathbf{B_3})$, then we can use $M = NG_1 = \mathrm{Ker} d_0^{G,1}$ as before even though it need not be free. The corresponding N is then isomorphic to $\mathrm{Ker} d_1^{G,1}$ with the bottom right hand corner being G_1. The 'construction data' is now replaced by data for killing some elements of $\pi_1(G)$, specified by $f_3 : \mathbf{B_3} \to M$. Although slightly at variance with the terminology used by Ellis, [9], we felt it sensible to introduce the term "totally free crossed square" for the type of free crossed square constructed in the above theorem, using "free crossed square" for the more general situation in which (M, G, ∂) and f_3 are specified and no requirement on (M, G, ∂) to be a free precrossed module is made.

3.3 The n-type of the k-skeleton

As in the other papers in this series, we will use the 'step-by-step' construction of a free simplicial group to observe the way in which the models react to the various steps of the construction.

In a 'step-by-step' construction of a free simplicial group, there are simplicial inclusions

$$\mathbf{F}^{(0)} \subseteq \mathbf{F}^{(1)} \subseteq \mathbf{F}^{(2)} \ldots$$

In general, considering the functor, $\mathfrak{M}(\quad, n)$, from the category of simplicial groups to that of crossed n-cubes, gives the corresponding morphisms

$$\mathfrak{M}(\mathbf{F}^{(0)},\ n) \to \mathfrak{M}(\mathbf{F}^{(1)},\ n) \to \mathfrak{M}(\mathbf{F}^{(2)},\ n) \to ... \to \mathfrak{M}(\mathbf{F},\ n).$$

We will investigate $\mathfrak{M}(\mathbf{F}^{(i)}, n)$, for $n = 0, 1, 2,$ and varying i.

Firstly look at $\mathfrak{M}(\mathbf{F}^{(0)}, n)$, where the 0-skeleton $\mathbf{F}^{(0)}$ can be thought of as simplifying to

$$\mathbf{F}^{(0)} : \quad \cdots \longrightarrow F(\mathbf{B_1}) \longrightarrow F(\mathbf{B_1}) \longrightarrow F(\mathbf{B_1})$$

with the $d_i^n = s_j^n =$ identity homomorphism on $F(\mathbf{B_1})$.

For $n = 0$, there is an equality

$$\mathfrak{M}(\mathbf{F}^{(0)}, 0) = F_0^{(0)}/d_1(\mathrm{Ker} d_0) = F(\mathbf{B_1}),$$

and so $\mathfrak{M}(\mathbf{F}^{(0)}, 0)$ is just the free group of 0-simplices of $\mathbf{F}$.

For $n = 1$, $\mathfrak{M}(\mathbf{F}^{(0)},1)$ is $NF_1^{(0)}/\partial_2 NF_2^{(0)} \longrightarrow F_0$. It is easy to show that $NF_1^{(0)}/\partial_2 NF_2^{(0)}$ is trivial and hence

$$\mathfrak{M}(\mathbf{F}^{(0)}, 1) \cong (1 \longrightarrow F(\mathbf{B_1})).$$

For $n = 2$, $\mathfrak{M}(\mathbf{F}^{(0)}, 2)$ is the trivial crossed square

$$\begin{array}{ccccc} NF_2/d_3^3(NF_3) & \longrightarrow & \text{Ker } d_0^1 & & 1 & \longrightarrow & 1 \\ \downarrow & & \downarrow & = & \downarrow & & \downarrow \\ \text{Ker } d_1^1 & \longrightarrow & F_1 & & 1 & \longrightarrow & F(\mathbf{B_1}). \end{array}$$

Next look at $\mathfrak{M}(\mathbf{F}^{(1)}, n)$ and recall that the 1-skeleton $\mathbf{F^{(1)}}$ is

$$\mathbf{F}^{(1)}: \quad ...\ F(s_1s_0(\mathbf{B_1}) \cup s_0(\mathbf{B_2}) \cup s_1(\mathbf{B_2})) \underset{s_1, s_0}{\overset{d_0, d_1, d_2}{\rightrightarrows}} F(s_0(\mathbf{B_1}) \cup \mathbf{B_2}) \underset{s_0}{\overset{d_1, d_0}{\rightrightarrows}} F(\mathbf{B_1}).$$

For $n = 0$, $\mathfrak{M}(\mathbf{F}^{(1)}, 0)$ is $F_0^{(1)}/d_1(\text{Ker}d_0) \cong F(\mathbf{B_1})/\partial_1 NF_1$, which is $\pi_0(\mathbf{F}^{(1)}) \cong \pi_0(\mathbf{F})$.

For $n = 1$, we have that

$$\begin{aligned} \mathfrak{M}(\mathbf{F}^{(1)}, 1) &= (NF_1/\partial_2 NF_2 \longrightarrow F_0), \\ &= \langle \mathbf{B_2} \rangle / P_1 \longrightarrow F(\mathbf{B_1}), \end{aligned}$$

which is a free crossed module. In fact this is the free crossed module on the (generalised) presentation $(\mathbf{B_1}; \mathbf{B_2}, f_2)$. As pointed out in [2], it is often convenient to generalise the notion of a presentation $(\mathbf{X}, \mathbf{R})$ with

$$\mathbf{R} \subset F(X)$$

to one with the map $\mathbf{R} \to F(X)$ specified and not necessarily monic. Thus if f_2 is injective, this is just a presentation $\mathcal{P}$ of $\pi_1(\mathbf{F})$. The kernel of this crossed module is then the module of identities of $\mathcal{P}$, again see [2].

For $n = 2$, $NF_2^{(1)} = \langle s_1(\mathbf{B_2}) \rangle \cap \langle Z \rangle$, so $\mathfrak{M}(\mathbf{F}^{(1)}, 2)$ simplifies to give (up to isomorphism),

$$\begin{array}{ccccc} NF_2/d_3^3(NF_3) & \longrightarrow & \text{Ker } d_0^1 & & J & \longrightarrow & \langle \mathbf{B_2} \rangle \\ \downarrow & & \downarrow & = & \downarrow & & \downarrow \\ \text{Ker } d_1^1 & \longrightarrow & G_1 & & \langle Z \rangle & \longrightarrow & F(s_0(\mathbf{B_1}) \cup \mathbf{B_2}) \end{array}$$

which is a crossed square with $J = (\langle s_1(\mathbf{B_2}) \rangle \cap \langle Z \rangle)/P_2$.

Next look at $\mathfrak{M}(\mathbf{F}^{(2)}, n)$. Recall the 2-skeleton $\mathbf{F}^{(2)}$ is

$$\mathbf{F}^{(2)}: \quad ...F(s_1s_0(\mathbf{B_1}) \cup s_0(\mathbf{B_2}) \cup s_1(\mathbf{B_2}) \cup \mathbf{B_3}) \underset{s_1, s_0}{\overset{d_0, d_1, d_2}{\rightrightarrows}} F(s_0(\mathbf{B_1}) \cup \mathbf{B_2}) \underset{s_0}{\overset{d_1, d_0}{\rightrightarrows}} F(\mathbf{B_1}).$$

The following can be easily obtained by direct calculation :

for $n = 0$,

$$\mathfrak{M}(\mathbf{F}^{(2)}, 0) = F_0/d_1(\mathrm{Ker} d_0) \cong \pi_0(\mathbf{F}^{(2)}) = \mathfrak{M}(\mathbf{F}^{(1)}, 0);$$

for $n = 1$,

$$\mathfrak{M}(\mathbf{F}^{(\mathbf{2})}, \mathbf{1}) \cong \langle \mathbf{B_2} \rangle / \mathbf{P_1} \longrightarrow \mathbf{F}(\mathbf{B_1}).$$

Finally, let $n = 2$. By an earlier result of this section, $\mathfrak{M}(\mathbf{F}^{(2)}, 2)$ corresponds to the free crossed square,

$$\begin{tikzcd} NF_2/d_3^3(NF_3) \arrow[r] \arrow[d] & \mathrm{Ker}\ d_0^1 \arrow[d] \\ \mathrm{Ker}\ d_1^1 \arrow[r] & F_1 \end{tikzcd} \quad = \quad \begin{tikzcd} J \arrow[r] \arrow[d] & \langle \mathbf{B_2} \rangle \arrow[d] \\ \langle Z_2 \rangle \arrow[r] & F(s_0(\mathbf{B_1}) \cup \mathbf{B_2}) \end{tikzcd}$$

where J is now $(\langle s_1(\mathbf{B_2}) \cup \mathbf{B_3} \rangle \cap \langle Z \cup \mathbf{B_3} \rangle)/P_2$ and $\langle Z_2 \rangle$ is $\langle Z \cup \mathbf{B_3} \rangle$, so this reduces to the earlier case if $\mathbf{B_3}$ is empty. Thus we have the following relations

$$\mathfrak{M}(\mathbf{F}^{(2)}, 0) = \mathfrak{M}(\mathbf{F}^{(1)}, 0), \qquad \mathfrak{M}(\mathbf{F}^{(2)}, 1) = \mathfrak{M}(\mathbf{F}^{(1)}, 1)$$

but $\mathfrak{M}(\mathbf{F}^{(2)}, 2)$ and $\mathfrak{M}(\mathbf{F}^{(3)}, 2)$ need not be the same due to the additional influence of $\mathbf{B_3}$. Of course it is clear that, in general:

$$\mathfrak{M}(\mathbf{F}^{(i)}, n) \quad = \quad \mathfrak{M}(\mathbf{F}^{(i+1)}, n) \quad \text{if} \quad i \geq n + 1.$$

4 Squared Complexes

The authors and Z. Arvasi have defined n-crossed complexes in [1]. In this paper, we will only need the case $n = 2$, which had already been defined by Ellis in [9]. We shall follow him in calling these *squared complexes*. A squared complex consists of a diagram of group homomorphisms

$$\begin{tikzcd} & & & N \arrow[dr, "\mu'"] & \\ \cdots \arrow[r] & C_4 \arrow[r, "\partial_4"] & C_3 \arrow[r, "\partial_3"] & L \arrow[u, "\lambda'"] \arrow[d, "\lambda"'] & P \\ & & & M \arrow[ur, "\mu"'] & \end{tikzcd}$$

together with actions of P on L, N, M and C_i for $i \geq 3$, and a function $h : M \times N \longrightarrow L$. The following axioms need to be satisfied.

(i) The square $\left(\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ M & \xrightarrow[\mu']{} & P \end{array}\right)$ is a crossed square;
(ii) The group C_n is abelian for $n \geq 3$;
(iii) The boundary homomorphisms satisfy $\partial_n \partial_{n+1} = 1$ for $n \geq 3$, and $\partial_3(C_3)$ lies in the intersection $\ker\lambda \cap \ker\lambda'$;
(iv) The action of P on C_n for $n \geq 3$ is such that μM and $\mu' N$ act trivially. Thus each C_n is a π_0-module with $\pi_0 = P/\mu M \mu' N$;
(v) The homomorphisms ∂_n are π_0-module homomorphisms for $n \geq 3$.

This last condition does make sense since the axioms for crossed squares imply that $\ker\mu' \cap \ker\mu$ is a π_0-module.

A morphism of squared complexes

$$\Phi : (C_*, \left(\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ M & \xrightarrow[\mu']{} & P \end{array}\right)) \longrightarrow (C'_*, \left(\begin{array}{ccc} L' & \xrightarrow{\lambda} & N' \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ M' & \xrightarrow[\mu']{} & P' \end{array}\right))$$

consists of a morphism of crossed squares $(\Phi_L, \Phi_N, \Phi_M, \Phi_P)$, together with a family of equivariant homomorphisms Φ_n for $n \geq 3$ satisfying $\Phi_L \partial_3 = \partial'_3 \Phi_L$ and $\Phi_{n-1}\partial_n = \partial'_n \Phi_n$ for $n \geq 4$. There is clearly a category $\mathfrak{SqComp}$ of squared complexes. This exists in both group and groupoid based versions.

By a *(totally) free squared complex*, we will mean one in which the crossed square is (totally) free, and in which each C_n is free as a π_0-module for $i \geq 3$.

Proposition 4.1 *There is a functor*

$$\mathcal{C}(\quad, 2) : \mathfrak{SimpGrp} \longrightarrow \mathfrak{SqComp}$$

such that free simplicial groups are sent to totally free squared complexes.

Proof:

Let $\mathbf{G}$ be a simplicial group or groupoid. We will define a squared complex $\mathcal{C}(\mathbf{G}, 2)$ by specifying $\mathcal{C}(\mathbf{G}, 2)_A$ for each $A \subseteq< 2 >$ and for $n \geq 3$, $\mathcal{C}(\mathbf{G}, 2)_n$. As usual, (cf. the other papers in this series, [17, 18, 19, 20]), we will denote by D_n the subgroup or subgroupoid of NG_n generated by the degenerate elements.

For $A \subset< 2 >$, we define

$$\mathcal{C}(\mathbf{G}, 2)_A = \mathfrak{M}(\mathfrak{sk}_2\mathbf{G}, 2)_A = \frac{\cap\{\mathrm{Ker} d_i^2 : i \in A\}}{d_3(\mathrm{Ker} d_0^3 \cap \bigcap\{\mathrm{Ker} d_{i+1}^3 : i \in A\} \cap D_3)}.$$

We do not need to define μ_i and the h-maps relative to these groups as they are already defined in the crossed square $\mathfrak{M}(\mathfrak{sk}_2\mathbf{G}, 2)$.

For $n \geq 3$, we set

$$\mathcal{C}(\mathbf{G}, 2)_n = \frac{NG_n}{(NG_n \cap D_n) d_{n+1}(NG_{n+1} \cap D_{n+1})}.$$

As this is part of the crossed complex associated to $\mathbf{G}$, we can take the structure maps to be those of that crossed complex, cf. [8, 19]. The terms are all modules over the corresponding π_0 as is easily checked. The final missing piece, ∂_3, of the structure is induced by the differential ∂_3 of NG.

The axioms for a squared complex can now be verified using the known results for crossed squares and for crossed complexes with a direct verification of those axioms relating to the interaction of the two parts of the structure, much as in [8] and [19].

Now suppose the simplicial group is free. The proof above of the freeness of $\mathfrak{M}(\mathfrak{sk}_2\mathbf{G},2)$ together with the freeness of the crossed complex of a free simplicial group, [19], now completes the proof. $\square$

Suppose that ρ is a general squared complex. The *homotopy groups* $\pi_n(\rho)$, $n \geq 0$ of ρ are defined cf. [9], to be the homology groups of the complex

$$\cdots \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_2} L \xrightarrow{\partial_2} M \rtimes N \xrightarrow{\partial_1} P \longrightarrow 1$$

with $\partial_2(l) = (\lambda' l^{-1}, \lambda l)$ and $\partial_1(m,n) = \mu(m)\mu'(n)$. The axioms of a crossed square guarantee that ∂_2 and ∂_1 are homomorphisms with $\partial_3(C_3)$ normal in $\mathrm{Ker}(\partial_2)$, $\partial_2(L)$ normal in $\mathrm{Ker}(\partial_1)$, and $\partial_1(M \rtimes N)$ normal in P.

Proposition 4.2 *The homotopy groups of* $\mathcal{C}(\mathbf{G},2)$ *are isomorphic to those of* $\mathbf{G}$ *itself.*

Proof:

Again this is a consequence of well-known results on the two parts of the structure. $\square$

5 Alternative Descriptions of Freeness.

In the context of CW-complexes, Ellis, [9] gave a neat description of the top group L in a (totally) free crossed square derived from that data. A simplicial group with a given CW-basis is the algebraic analogue of a CW-complex so one would expect a similar result to hold in that setting. Ellis uses the generalised van Kampen theorem of Brown and Loday, [3]. In the algebraic setting no such tool is available, but in fact its use is not needed.

Ellis' description is in terms of tensor products and coproducts. For completeness we recall the background definitions of these constructions.

5.1 Tensor Products

Suppose that $\mu : M \to P$ and $\nu : N \to P$ are crossed modules over P. The groups M and N act on each other, and themselves, via the action of P.

The tensor product $M \otimes N$ is the group generated by the symbols $m \otimes n$ for $m \in M$, $n \in N$ subject to the relations

$$mm' \otimes n = ({}^{m}m' \otimes {}^{m}n)(m \otimes n),$$

$$m \otimes nn' = (m \otimes n)({}^{n}m \otimes {}^{n}n'),$$

for $m, m' \in M,\ n, n' \in N$. There are homomorphisms $\lambda : M \otimes N \to M,\ \lambda' : M \otimes N \to N$ defined on generators by $\lambda(m \otimes n) = m({}^{n}m)^{-1}$ and $\lambda'(m \otimes n) = ({}^{m}n)n^{-1}$. The group P acts on $M \otimes N$ by ${}^{p}(m \otimes n) = ({}^{p}m \otimes {}^{p}n)$, and there is a function $h : M \times N \to M \otimes N$, $(m, n) \longmapsto m \otimes n$. In [3], it is verified that this structure gives a crossed square

$$\begin{tikzcd} M \otimes N \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd}$$

with the universal property of extending the corner

$$\begin{tikzcd} & N \arrow[d, "\nu"] \\ M \arrow[r, "\mu"'] & P \end{tikzcd} .$$

5.2 Coproducts

Let $(M, P, \partial_1), (N, P, \partial_2)$ be P-crossed modules. Then N acts on M, and M acts on N, via the given actions of P. Let $M \rtimes N$ denote the semidirect product with the multiplication given by

$$(m, n)(m', n') = (mm',\ {}^{m'}nn')$$

and injections

$$\begin{array}{rclcrl} i' : M \to M \rtimes N & & \text{and} & & j' : N \to M \rtimes N \\ m \longmapsto (m, 1) & & & & n \longmapsto (1, n). \end{array}$$

We define the pre-crossed module

$$\begin{array}{c} \underline{\delta} : M \rtimes N \to P \\ (m, n) \longmapsto \partial_1(m)\partial_2(n). \end{array}$$

Let $\{M, N\}$ be the subgroup of $M \rtimes N$ generated by the elements of the form

$$(m^{n}m^{-1}, n^{m}n^{-1})$$

for all $m \in M$, $n \in N$, thus we are able to form the quotient group $M \rtimes N/\{M, N\}$ and obtain an induced morphism

$$\partial : M \rtimes N/\{M, N\} \to P$$

given by

$$\partial(m, n)\{M, N\} = \partial_1(m)\partial_2(n).$$

Let $q : M \rtimes N \to M \rtimes N/\{M, N\}$ be projection and let $i = qi'$, $j = qj'$. Then $M \circ N = (M \rtimes N)/\{M, N\}$ with the morphisms i, j, is *a coproduct* of (M, P, ∂_1) and (N, P, ∂_2) in the category of P-crossed modules.

Proposition 5.1 *[9] Let $(L, M, \bar{M}, M \rtimes F)$ be a (totally) free crossed square on the 2-dimensional construction data or on functions (f_2, f_3) as described above. Let $\partial : C \to M \rtimes F$ be the free crossed module on the function $\mathbf{B_3} \to M \rtimes F$ given by $y \longmapsto (f_3 y, 1)$. From the crossed module $M \otimes \bar{M} \to M \rtimes F$, then L is isomorphic to the coproduct $(M \otimes \bar{M}) \circ C$ factored by the relations*

$$\begin{array}{ll} 1) & i(\partial c \otimes \bar{m}) = j(c)j({}^{\bar{m}}c^{-1}) \\ 2) & i(m \otimes \partial c) = j({}^{m}c)j(c^{-1}) \end{array}$$

for $c \in C$, $m \in M$ and $\bar{m} \in \bar{M}$.

The homomorphisms $L \to M$, $L \to \bar{M}$ are given by the homomorphisms

$$\lambda : M \otimes \bar{M} \to M \qquad \textit{and} \qquad \lambda' : M \otimes \bar{M} \to \bar{M}$$

and $\partial : C \to M \cap \bar{M}$. The h-map of the crossed square is given by

$$h(m, \bar{n}) = i(m \otimes \bar{n})$$

for $m, n \in M$.

Proof: This comes by direct verification using the universal properties of tensors and coproducts. □

Remark: For future applications it is again important to note that the result is not dependent on the crossed square being *totally* free, although this is the form proved and used by Ellis, [9]. If $M \to F$ is any pre-crossed module, one can form the 'corner'

$$\begin{tikzcd} & M \arrow[d] \\ \bar{M} \arrow[r] & M \rtimes F, \end{tikzcd}$$

complete it to a crossed square via $M \otimes \bar{M}$ and then add in $\mathbf{B_3} \to M$. Nowhere does this use freeness of $M \to F$.

Corollary 5.2 *Let $\mathbf{G^{(1)}}$ be the 1-skeleton of a simplicial group. Then in the free crossed square $\mathfrak{M}(\mathbf{G^{(1)}}, \mathbf{2})$ described above,*

$$NG_2^{(1)}/\partial_3 NG_3^{(1)} \cong Kerd_1^1 \otimes Kerd_0^1.$$

Proof: This is clear from the previous proposition. □

Remark:

If we set $M = \mathrm{Ker}d_0^1 = NG_1^{(1)}$, then the identification given by the Corollary gives

$$NG_2^{(1)}/\partial_3 NG_3^{(1)} \cong M \otimes \bar{M}.$$

This uses the fact that $\mathrm{Ker}d_0^1$ and $\mathrm{Ker}d_1^1$ are linked via the map sending m to $ms_0d_1m^{-1}$ for $m \in \mathrm{Ker}d_0^1$. The h-map $h : M \times \bar{M} \to NG_2^{(1)}/d_3^3 NG_3^{(1)}$ is $h(x, y) = [s_1x, s_1ys_0y^{-1}]d_3^3 NG_3^{(1)}$, but this is also $h(x, y) = x \otimes y$. Thus

$$x \otimes y = [s_1x, s_1ys_0y^{-1}]d_3^3 NG_3^{(1)}$$

under the identification via the isomorphism of 5.2.

This explains the 'mysterious' formula of [17] in the discussion before Proposition 4.6 of that paper.

5.3 Applications to 2-crossed complexes.

Of course there are similar results for free squared complexes. What is less obvious is the way in which these results can be applied to the situation that we studied in our earlier paper, [20]. There we considered the alternative model for 3-types given by Conduché's 2-crossed modules and also looked at the corresponding 2-crossed complexes. We will not repeat all that discussion here but note the definition:

Definition:

A 2-crossed complex of group(oid)s is a sequence of group(oid)s

$$C : \qquad \ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which

(i) C_n is abelian for $n \geq 3$;

(ii) C_0 acts on C_n, $n \geq 1$, the action of ∂C_1 being trivial on C_n for $n \geq 3$;

(iii) each ∂_n is a C_0-group(oid) homomorphism and $\partial_i\partial_{i+1} = 1$ for all $i \geq 1$; and

(iv) $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ is a 2-crossed module.

We refer the reader to [5] or [20] for the exact meaning of 2-crossed module.

Given a simplicial group or groupoid, $\mathbf{G}$, define

$$C_n = \begin{cases} NG_n & \text{for } n = 0, 1 \\ NG_2/d_3(NG_3 \cap D_3) & \text{for } n = 2 \\ NG_n/(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1}) & \text{for } n \geq 3 \end{cases}$$

with ∂_n induced by the differential of $\mathbf{NG}$. Note that the bottom three terms (for $n = 0$, 1, and 2) form a 2-crossed module considered in [5] or [20] and that for $n \geq 3$, the groups are all $\pi_0(G)$-modules, since in these dimensions C_n is the same as the corresponding crossed complex term (cf. Ehlers and Porter [8] for instance).

Proposition 5.3 *[20]*
With the above structure (C_n, ∂_n) is a 2-crossed complex, which will be denoted $C(\mathbf{G})$. □

Here we note in particular that the term C_2 is $NG_2/d_3(NG_3 \cap D_3)$ and so is the same as $\mathcal{C}(\mathbf{G}, 2)_{<2>}$. Thus if $\mathbf{G}$ is a simplicial group, we obtain *gratis*:

Corollary 5.4 *Let $\mathbf{G^{(1)}}$ be the* 1*-skeleton of a simplicial group. The 2-crossed complex of $\mathbf{G^{(1)}}$ satisfies*

$$C(\mathbf{G}^{(1)})_2 \cong Kerd_1^1 \otimes Kerd_0^1.$$

□

We also get in general a description of $C(\mathbf{G^{(2)}})_\mathbf{2}$ as a quotient of the form $(\mathrm{Ker}d_1^1 \otimes \mathrm{Ker}d_0^1 \circ C)/\sim$ where as in Proposition 5.1, this C is a free crossed module on the 'new cells' in dimension 2.

5.4 The suspension of a $K(\pi, 1)$.

As was mentioned in [19], Brown and Loday used their generalised van Kampen Theorem, [3], to calculate $\pi_3 \Sigma K(\pi, 1)$ for π a group, as the kernel of the commutator map from $\pi \otimes \pi$ to π. Jie Wu, ([22] Theorem 5.9), for any group π and set of generators $\{x_\alpha | \alpha \in J\}$ for π, gives a presentation of $\pi_n \Sigma K(\pi, 1)$ in terms of higher commutators, but does not manage to get the Brown-Loday result explicitly although his result is clearly linked to theirs.

Wu's methods use a study of simplicial groups and a construction he ascribes to Carlsson, [4]. This gives a simplicial group $F^\pi(S^1)$ that has $\pi_{n+2}\Sigma K(\pi, 1) \cong \Omega\Sigma K(\pi, 1) \cong \pi^{n+1} F^\pi(S^1)$. As we pointed out in [17], $F^\pi(S^1)$ is a pointed analogue of the 'tensorisation' of $K(\pi, 0)$, the constant simplicial group on π, with the simplicial circle S^1. In general if G is a simplicial group and K a pointed simplicial set, $G\bar{\wedge}K$ will denote the simplicial group with group of n- simplices given by

$$\coprod_{x \in K_n} (G_n)_x/(G_n)_*.$$

If $x \in K_n$, we denote the x-indexed copy of $g \in G_n$ within $(G\bar{\wedge}K)_n$ by $g\bar{\wedge}x$. The face and degeneracy maps of $G\bar{\wedge}K$ are induced by the componentwise application of the corresponding morphisms of G and K

$$d_i(g\bar{\wedge}x) = d_i^G g \bar{\wedge} d_i^K x,$$

$$s_i(g\bar{\wedge}x) = s_i^G g\bar{\wedge}s_i^K x.$$

Of course if $d_i^K x = *$ then $d_i(g\bar{\wedge}x) = 1$.

The case of interest to us is $G = K(\pi, 0)$, $K = S^1$ and we will adopt the notation for simplices in S^1 used by us in [17]. We write $S_0^1 = \{*\}$ and will take $*$ to denote the corresponding degenerate n-simplex basing S_n^1 in all dimensions; $S_1^1 = \{\sigma, *\}$, $S_2^1 = \{x_0, x_1, *\}$, where $x_0 = s_1\sigma$, $x_1 = s_0\sigma$ and in general $S_{n+1}^1 = \{x_0, \ldots , x_n, *\}$, where $x_i = s_n \ldots s_{i+1}s_{i-1} \ldots s_0\sigma$, $0 \leq i \leq n$.

We write $G = K(\pi, 0)$ for simplicity and will usually make no distinction between simplices in different dimensions unless confusion might arise. We have

$(G\bar{\wedge}S^1)_0 = 1$, the trivial group,

$(G\bar{\wedge}S^1)_1 \cong \pi$,

$(G\bar{\wedge}S^1)_2 \cong \pi * \pi$, the free product of two copies of π, and so on.

The group $(G\bar{\wedge}S^1)_n$ is a free product of n-copies of π, $\coprod\{(\pi)_x : x \in S_n^1\backslash\{*\}\}$, and writing as above $g\bar{\wedge}x$ for the x-indexed copy of $g \in \pi$ in this, we note that $(g\bar{\wedge}x)(g'\bar{\wedge}x) = (gg'\bar{\wedge}x)$ for g, $g' \in \pi$. As $g\bar{\wedge}x_i^{(n+1)} = s_n(g\bar{\wedge}x_i^{(n)})$ holds in all dimensions, $n \geq 2$ and for all $0 \leq i \leq n$, it is clear that $N(G\bar{\wedge}S^1)_n = D_n$, that is, it is generated by degenerate elements in all dimensions $n \geq 2$, we can therefore apply Corollary 5.2. As $N(G\bar{\wedge}S^1)_0$ is trivial, $\mathrm{Ker}d_0^1 = \mathrm{Ker}d_1^1 = (G\bar{\wedge}S^1)_1 \cong \pi$, so we get:

For $H = G\bar{\wedge}S^1$,

$$NH_2/\partial_3 NH_3 \cong \pi \otimes \pi.$$

We have by [21] that the algebraic 2-type of H is completely modelled by the crossed square $\mathfrak{M}(H, 2)$, that is by

$$\begin{tikzcd} \pi \otimes \pi \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & \pi \arrow[d, "="] \\ \pi \arrow[r, "="'] & \pi, \end{tikzcd}$$

where μ_1 and μ_2 are the commutator maps.

As a consequence we have:

Corollary 5.5 *The 3-type of $\Sigma K(\pi, 1)$ is completely specified by the above crossed square. In particular there is an isomorphism*

$$\pi_3(\Sigma K(\pi, 1)) \cong Ker(\mu : \pi \otimes \pi \to \pi).$$

□

This result was first found by Brown and Loday [3]. Their proof was an illustration of the use of their generalised van Kampen Theorem. Jie Wu, [22], gives some methods that shed light on the higher homotopy groups, but although they yield a description of π_4, they do not analyse the 4-type itself.

The model $G\bar{\wedge}S^1$ is 1-skeletal and one might expect that a triple tensor $\pi \otimes \pi \otimes \pi$ may be involved in any model of its 4-type. Of course $\mathfrak{M}(H,3)$ gives a complete model, but the individual terms involved in that model are not as easy to analyse as in $\mathfrak{M}(H,2)$. An amalgam of Wu's methods and the methods developed in the earlier papers of this series, [17, 18, 19, 20], might provide insight into this. This problem is not of itself that important, but it does seem to provide an excellent testbed for the development of methods to aid in calculation with low dimensional algebraic models of homotopy types.

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